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Self-avoiding polygons on the square, L and Manhattan lattices

I G Enting† and A J Guttmann‡

† Division of Atmospheric Research, CSIRO, Aspendale, Vic. 3195, Australia
‡ Department of Mathematics, Statistics and Computer Science, The University of Newcastle, New South Wales, 2308, Australia

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Abstract. Transfer-matrix techniques are used to extend the self-avoiding polygon generating function on the square lattice to terms in x^{46}, corresponding to 46 step polygons. These techniques are then extended to apply to directed square lattices, such as the L and Manhattan lattice, and the self-avoiding polygon generating function to x^{48} is found for these lattices.

Series analysis confirms that the 'specific heat' exponent $\alpha = \frac{1}{2}$ for the self-avoiding walk problem, and gives the following estimates for the connective constants: $\mu(\mathrm{SQ}) = 2.638\,155 \pm 0.000\,025$, $\mu(\mathrm{L}) = 1.5657 \pm 0.0019$ and $\mu(\mathrm{Man.}) = 1.7328 \pm 0.0005$. Some evidence for a correction to scaling exponent $\Delta \approx 0.84$ is found from square lattice series.

1. Introduction

In an earlier paper (Enting 1980) it was shown how generating function techniques could be used to extend the known polygon generating function series for the square lattice, and the series was obtained to x^{38}. In this paper we extend that calculation to x^{46}, and show how the method can be applied to directed lattices, such as the L lattice (a square lattice on which each step must be perpendicular to its predecessor) and the Manhattan lattice, in which adjacent rows (columns) have antiparallel directions, corresponding to the traffic pattern in Manhattan. These lattices are shown in figure 1. The method and its extension are discussed in § 2.

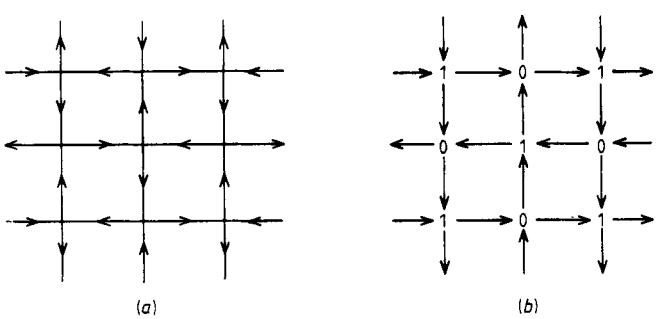

Figure 1. (*a*) L lattice orientations. (*b*) Manhattan lattice showing orientations and the two distinct types of site.

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These extended series are analysed in § 3. For the square lattice in particular it has been possible to obtain a very accurate estimate of the connective constant, with an uncertainty of 1 part in 10^5. For the L and Manhattan lattices less accurate estimates of the connective constant are obtained.

2. Enumerating polygons

The enumeration techniques that we used are based on those used by Enting (1980) in counting square lattice polygons of up to 38 steps. There are two separate combinatorial aspects involved in this approach. The first problem is to enumerate all polygons of a particular class that can be contained within a rectangle of a given size. The second problem is to construct a linear combination of these finite-rectangle enumerations that gives the correct enumeration for the infinite lattice.

The enumerations for the finite rectangles are performed using transfer matrix techniques, based on the concept of building up a rectangular array one site at a time. In order to perform the enumeration as efficiently as possible, the enumerations are performed for rectangles whose width is not greater than their length and the construction is based on taking each particular width and adding columns of sites so as to extend the length. This approach minimises the size of the vectors on which the transfer matrices operate. The enumerations for rectangles whose width exceeds their length is obtained using the symmetry of the lattice. For each rectangle, what is counted is the number of polygons that span the full length of the rectangle and which obey any special constraints for the particular lattice. The polygons are not required to span the full width of the rectangles. This simplifies the transfer matrix method of enumeration but it does introduce an asymmetry that must be considered when reconstructing the final infinite lattice enumeration.

In enumerating the polygons, the 'self-avoiding' constraint and the special constraints on the directed lattice are 'local' constraints that are readily embodied in a transfer matrix that builds on one lattice site and up to two steps of the polygon at each iteration. There is also a 'global' requirement that the resulting graphs consist of a single component. This can be ensured by

(i) requiring that all graphs span the rectangle so that it is impossible to generate a sequence of disjoint polygons along the length of the rectangle;

(ii) specifying the connectivities of the free ends of the graphs as they are generated so that it is forbidden to complete any single polygon component if the partially constructed graph has free ends from which another disconnected component can be constructed.

In order to specify the connectivities of the loops of the partially constructed polygons, it is sufficient to label the ends with 1 or 2 according to whether the end is the first or second end of the loop encountered when traversing the width of the rectangle. The various possible combinations are shown in figure 6 of Enting (1980) and these combinations lead to the rules shown in figure 3 of this paper. These rules embody transformations of the original problems. For the L lattice, we count square lattice polygons that change direction at every site. On the Manhattan lattice we count square lattice polygons restricted so that, on half the sites, the vertices are one of the four forms shown in figure 2(a) while on the other half, the vertices are one of the four types shown in figure 2(b).

Figure 2. (*a*) The four types of junction allowed at type 0 sites on Manhattan lattice polygons. (*b*) The junctions allowed at type 1 sites of Manhattan lattice polygons.

To show the equivalence in the L lattice case, every L lattice polygon corresponds to one possible square lattice polygon with a turn at each vertex, and every square lattice polygon that does have a turn at each vertex can be oriented consistently with the L lattice by following the orientation from any one of its steps.

In the Manhattan lattice case, the argument is similar in that every Manhattan lattice polygon is a square lattice polygon and will have its sites alternating between the types shown in figures 2(*a*) and (*b*). Any square lattice graph that obeys these constraints can be oriented in a manner consistent with the Manhattan lattice by taking the orientations of the pair of edges that meet at any right angle turn, so long as the two types of site shown in figure 2 are assigned in a manner consistent with the arrows as in figure 1. Formally at least, it is necessary to construct two sets of rectangles in the Manhattan lattice enumeration, those starting with type 0 sites and those starting with type 1 sites. As the rectangles are built up it is necessary to keep track of the type of the site that is currently being added so that the rules shown in figure 3 can be applied.

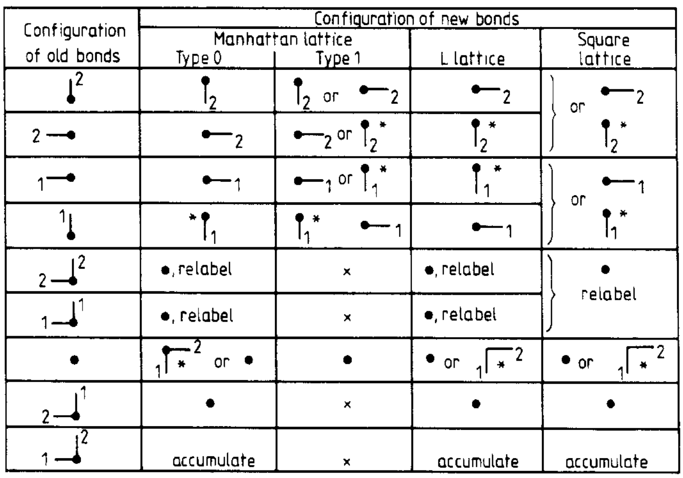

Figure 3. The configuration of edges that can be built onto a site, given the various possible existing edges at that site. × denotes an input which is either not possible in that case or for which there is no allowed continuation. 'Accumulate' means that this configuration cannot contribute to larger rectangles but will contribute for the current length if the partial graph has no free ends. '*related*': this junction will have changed the loop connectivity and other edges must be relabelled. The labels 1, 2 define the connectivity (see Enting 1980). * denotes configurations that are forbidden in the bottom row of a rectangle.

The transfer matrix formalism enumerates the polygons by building up vectors of generating functions. A line is drawn across the width of the rectangle as shown in figure 7 of Enting (1980). Each possible way in which the loop edges can cross this line, and each possible labelling of these intersections with 1 and 2, is assigned to a distinct vector component. The combinations in figure 3 show which new sets of intersections become possible when a single site (shown as ●) is added. These new configurations make a contribution to the new vector component that is equal to the 'old' component multiplied by x to the power of the number of new edges (i.e. x^0, x^1 or x^2). Detailed examples of the use of generating functions are given by Enting (1980). The last column of table 2 of Enting (1980) gives the number of vector components needed for various widths of rectangles. (Note that the widths specified in that table are numbers of sites, not numbers of steps as used below.)

For the second part of the procedure, the basic combinatorics for combining generating functions for finite rectangles was given by Enting (1980). If $g_{mn}(x)$ is the generating function for all polygons that fit into an $m \times n$ rectangle but not into any smaller rectangle, then the generating function for the number of square lattice polygons per lattice site is

$$U(x) = \sum_{m,n} g_{mn}(x). \tag{2.1}$$

This can be approximated by

$$U(x) \approx U_k(x) = \sum_{\substack{m,n \\ m+n<k}} g_{mn}(x) \tag{2.2}$$

which will correctly enumerate all polygons of up to $2k$ steps. This approximation can be rewritten as

$$U_{2p+1}(x) = \sum_{\substack{m \leqslant p \\ n:\, m+n \leqslant 2p+1}} a_{mn} g_{mn} \tag{2.3}$$

where

$$a_{mm} = 1, \qquad m \leqslant p, \tag{2.4a}$$

$$a_{mn} = 2, \qquad m < n,\ m+n \leqslant 2p+1, \tag{2.4b}$$

$$a_{mn} = 0, \qquad \text{otherwise.} \tag{2.4c}$$

The transfer matrix formalism defined above actually calculates $G_{mn}(x)$ which is the generating function for polygons that fit into an $m \times n$ rectangle but not into any $m \times p$ rectangle for $p < n$. Thus

$$G_{pn} = \sum_{m \leqslant p} (p - m + 1) g_{mn}. \tag{2.5}$$

This can be inverted to give

$$g_{mn} = G_{mn} - 2G_{m-1,n} + G_{m-2,n}. \tag{2.6}$$

Substituting into (2.3) gives

$$U_{2p+1}(x) = \sum_{\substack{m \leqslant p \\ n:\, m+n \leqslant 2p+1}} C_{mn} G_{mn} \tag{2.7}$$

where

$$C_{mm} = 1, \qquad m \leq p, \tag{2.8a}$$

$$C_{m-2,m} = -1, \qquad m \leq p, \tag{2.8b}$$

$$C_{m,n} = 2, \qquad m \leq n,\ m+n = 2p+1, \tag{2.8c}$$

$$C_{m,n} = -2, \qquad m < n,\ m+n = 2p, \tag{2.8d}$$

$$C_{m,n} = 0, \qquad \text{otherwise.} \tag{2.8e}$$

(Note that the condition $m < n$ in (2.8d) was incorrectly given as $m \leq n$ by Enting (1980).)

Expression (2.7) will correctly enumerate square lattice polygons of up to $4p+2$ steps. For the L lattice all polygons have $4n$ steps and so the largest polygons correctly enumerated by (2.7) are those of $4p$ steps. However, on the L lattice, expression (2.7) also enumerates all but $(p-1)$ of the polygons of $(4p+4)$ steps and so the series can be extended by applying the simple correction of $p-1$. The $p-1$ graphs that have to be added explicitly are most easily described in terms of square lattice graphs derived from them by linking every second site to give a polygon of $2p+2$ steps. These square lattice graphs are simple rectangles. Each L lattice graph corresponds to two square lattice graphs and each square lattice rectangle corresponds to two L lattice graphs, except for the $1 \times p$ and $p \times 1$ rectangles which each give only one L lattice polygon. Thus p possible rectangles give $p-1$ distinct L lattice graphs that fit into a $(p+1) \times (p+1)$ square on the L lattice but not into any smaller rectangle. We have explicitly checked this argument by applying our procedure for evaluating (2.7) using various values of p and comparing the results with our final series. In each case the $(4p+4)$th term was $(p-1)$ less than the term obtained using larger values of p.

A different combinatorial scheme was used on the Manhattan lattice, firstly because of the need to distinguish type 0 and 1 sites and secondly to make use of the fact that the $g^{(0)}_{mn}$ and $g^{(1)}_{mn}$ (generalisations of g_{mn}, labelled according to the type of the top left site) are zero unless both m and n are odd. If G_{mn} is also generalised to $G^{(0)}_{mn}$ and $G^{(1)}_{mn}$ labelled according to the type of the top left site then

$$U_k(x) = \sum_{m+n \leq k} a_{mn}(g^{(0)}_{mn} + g^{(1)}_{mn}). \tag{2.9}$$

Because both types of rectangle are included, $U_k(x)$ thus generates the number of L lattice polygons per two lattice sites.

Equation (7.5) generalises to

$$G^{(0)}_{mn} + G^{(1)}_{mn} = \sum_{p \leq ra} (m-p+1)(g^{(0)}_{pn} + g^{(1)}_{pn}). \tag{2.10}$$

Inverting (2.10) gives

$$g^{(0)}_{pn} + g^{(1)}_{pn} = \sum_{\text{odd } m \leq p} \tilde{C}_{pm}(G^{(0)}_{mn} + G^{(1)}_{mn}), \qquad p, n \text{ odd}, \tag{2.11}$$

with

$$\tilde{C}_{pp} = 1, \qquad \tilde{C}_{p,p-2} = -3, \tag{2.12a, b}$$

$$\tilde{C}_{p,p-2j} = (-1)^j 4, \qquad 2 \leq j < \tfrac{1}{2}p. \tag{2.12c}$$

Because only odd m, n have non-zero g_{mn}, if the G_{mn} can be evaluated up to some

maximum width p, expression (2.9) is most efficiently used in the form

$$U_{2p+2}(x) = \sum_{\substack{\text{odd } m \leqslant p \\ n: m+n \leqslant 2p+2 \\ n \text{ odd}}} a_{mn}(g^{(0)}_{mn} + g^{(1)}_{mn}), \qquad p \text{ odd}, \tag{2.13}$$

which correctly enumerates polygons of up to $(4p+4)$ steps. Substituting (2.11) into (2.13) gives

$$U_{2p+2}(x) = \sum d_{mn}(G^{(0)}_{mn} + G^{(1)}_{mn}), \qquad p \text{ odd}, \tag{2.14}$$

with

$$d_{mm} = 1, \qquad \text{odd } m \leqslant p, \tag{2.15a}$$

$$d_{m-2,m} = -1, \qquad \text{odd } m \leqslant p, \tag{2.15b}$$

$$d_{m,2p+2-m} = 2, \qquad \text{odd } m \leqslant p, \tag{2.15c}$$

$$d_{m,2p+2-m-2j} = (-1)^j 4, \qquad \text{odd } m \leqslant p, j \geqslant 1, 2p+2-m-2j \geqslant 1, \tag{2.15d}$$

$$d_{m,n} = 0, \qquad \text{otherwise}. \tag{2.15e}$$

The calculations were performed using the residue arithmetic of integers modulo various primes. To reconstruct the counts required 4, 2 and 2 primes for square, L and Manhattan lattices respectively. (The primes used were the largest successive primes less than 2^{15}.) In each case the maximum width (i.e. the quantity p in expressions (2.7) and (2.13)) was 11 steps, which meant that 418 35 vector components were required (Enting 1980) and that the series were obtained correctly to 46, 48 and 48 step polygons for the square, L and Manhattan lattices. As discussed above, to enumerate the 48 step polygons on the L lattice it was necessary to add 10 to the enumeration obtained by evaluating expression (2.13) with $p = 11$.

The calculations were performed using a Perkin-Elmer 3220 minicomputer at the University of Newcastle running Unix level 7. Approximately 5 days was required for each prime for the square lattice and 1 day for the L lattice calculations. The Manhattan lattice run took 2 days because both $G^{(0)}_{mn}$ and $G^{(1)}_{mn}$ were calculated. These times are a reflection of the limited hardware at our disposal, more than 60% of the time being taken by disc access. With a virtual memory operating system we would expect the times to be at least halved.

The times taken for the directed lattice appear relatively long, considering the small numbers of graphs that are actually enumerated. These could have been reduced if the programs had been modified to take account of the fact that all the directed lattice polygons have $4n$ steps. In addition it must be remembered that the amount of computation required to enumerate polygons by direct construction tends to grow as the number of self-avoiding walks rather than as the number of polygons. This type of behaviour would suggest that the transfer matrix approach may not be the most efficient way of enumerating directed lattice polygons—we chose this method because it involved only minor modifications to existing programs. We have not been able to find a way of exploiting the relation between L lattice polygons of $4n$ steps and square lattice polygons of $2n$ steps. In order to apply this relation, additional information is required in the square lattice enumeration. This increases the size of the vectors of partial generating functions so that they are essentially as large as the vectors required when enumerating L lattice polygons by the techniques described above.

The resulting polygon counts are shown in table 1.

Table 1. Coefficients of polygon generating function of the square, Manhattan and L lattices.

n	Square	Manhattan	L
4	1	1	1
6	2	0	0
8	7	2	0
10	28	0	0
12	124	7	1
14	588	0	0
16	2 938	32	2
18	15 268	0	0
20	81 826	168	9
22	449 572	0	0
24	2 521 270	970	36
26	14 385 376	0	0
28	83 290 424	5 984	154
30	488 384 528	0	0
32	2 895 432 660	38 786	684
34	17 332 874 364	0	0
36	104 653 427 012	261 160	3 128
38	636 737 003 384	0	0
40	3 900 770 002 646	1 812 630	14 666
42	24 045 500 114 388	0	0
44	149 059 814 328 236	12 895 360	70 258
46	928 782 423 033 008	0	0
48		93 638 634	342 766

3. Analysis of series

It follows from Nienhuis's (1982, 1984) results and scaling laws that the polygon generating function (PGF) should have a cusp-like singularity with an exponent of $\frac{3}{2}$. That is, we expect

$$P(x) = \sum_{n \geqslant 1} p_{2n} x^{2n} \sim A(x) + B(x)(1-\mu^2 x^2)^{2-\alpha} + C(x)(1-\mu^2 x^2)^{2-\alpha+\Delta} \tag{3.1}$$

where $\alpha = \frac{1}{2}$, and A, B and C are even functions of x regular in the physical disc $|x^2| \leqslant \mu^{-2}$. $\Delta \neq 1$ is a confluent correction exponent and P is an even function of x due to the loose-packed lattice structure. Initially, we will confine our analysis to the square lattice PGF.

Padé approximants cannot be used on (3.1) directly, due to their inability to handle cusp-like singularities (see e.g. Gaunt and Guttmann 1974). Differentiating the series three times sharpens the physical singularity, so it now diverges at $x^2 = \mu^{-2}$ with an exponent of $(-1-\alpha)$. Unfortunately, dlog Padés are only slowly convergent, and enable us to estimate only $\mu^{-2} = 0.1437 \pm 0.0001$ with $1+\alpha = 1.51 \pm 0.02$.

The recurrence relation method (Guttmann and Joyce 1972) can be used for cusp-like singularities such as (3.1). The second-order recurrence relations can accommodate the first two terms in (3.1), while third-order recurrence relations can additionally accommodate any confluent term. The estimates of μ^2 and $2-\alpha$ as found by this method are shown in table 2. Rapid convergence is obtained, and we make the estimates $\mu^{-2} \approx 0.143\,68 \pm 0.0001$, $2-\alpha = 1.500 \pm 0.005$.

Table 2. Analysis of square lattice polygon generating function by recurrence relation method. n is the degree of the polynomial multiplying each coefficient in the recurrence method.

	2nd order		3rd order	
n	μ^{-2}	$2-\alpha$	μ^{-2}	$2-\alpha$
2	—	—	0.143 006	1.6616
3	0.143 031	1.6401	0.143 658	1.5070
4	0.143 895	1.4250	0.143 683	1.4986
5	0.143 672	1.5048		
6	0.143 681	1.5003		

Both unbiased methods therefore support the result $\alpha = \frac{1}{2}$. If we plot estimates of μ^{-2} against estimates of $2-\alpha$ from the recurrence relation method results, we obtain a seemingly linear relationship, just as found for pole-residue plots in the Padé method. In this way, we estimate that if $\alpha = \frac{1}{2}$, $\mu^{-2} = 0.143\,6810$ or $\mu = 2.638\,155$. Our earlier estimate (Guttmann 1984), based on an analysis of the chain generating function, was $\mu = 2.6381 \pm 0.0002$, in excellent agreement with the above, while Berretti and Sokal (1985) have found $2.638\,20 \pm 0.000\,34$ from Monte Carlo analysis.

An alternative unbiased extrapolation method that appears to be very effective is the Levin u-transform. This appears to have been first used in a similar context by Barber *et al* (1984), and is discussed at some length by Smith and Ford (1979), who have reviewed 11 different 'standard' numerical methods for sequence extrapolation, and found the Levin transform the most successful general purpose method. In that method, if we wish to extrapolate a sequence $\{a_n^{(0)}\}_{n=0}^{N}$, we do so by forming sequences $\{a_n^{(k)}\}_{n=0}^{N-2k}$, $k = 1, 2, 3, \ldots$, defined by

$$a_n^{(k)} = \Delta^k(a_n^{(0)} c_n^{(k)})/\Delta^k c_n^{(k)}, \qquad c_n^{(k)} = n^{k-1}/n\Delta a_n^{(0)}, \tag{3.2}$$

where

$$\Delta y_n = y_{n+1} - y_n.$$

In our case we take $a_n^{(0)} = p_{2n}/p_{2n-2}$, the ratio of successive coefficients in (3.1). The results are shown in table 3.

The sequence $\{a_n^{(2)}\}$ is decreasing, while $\{a_n^{(3)}\}$ is increasing. The last entries in each case appear to provide 'bounds' so that $6.9593 < \mu^2 < 6.9602$ or $\mu = 2.6381 \pm 0.0001$ in precise agreement with our biased analysis of the SAW series.

The Levin transform method can also be used to provide biased estimates as follows. Since we expect the ratios $p_{2n}/p_{2n-2} \sim \mu^2[1 + (\alpha - 3)/n]$, we take $a_n^{(0)} = (p_{2n}/p_{2n-2})/[1 + (\alpha - 3)/n]$. Subsequent sequences are also shown in table 3. The elements of $\{a_n^{(2)}\}$ are increasing, but the rate of increase is rapidly slowing. The last entries of $\{a_n^{(3)}\}$ are stable at $\mu^2 = 6.959\,86$, and we estimate $\mu^2 = 6.959\,86 \pm 0.000\,13$ or $\mu = 2.638\,155 \pm 0.000\,025$, in precise agreement with the recurrence relation method results previously quoted, and right in the middle of the current series and Monte Carlo estimates. The phenomenological renormalisation scheme of Derrida (1981) gives $\mu = 2.638\,17 \pm 0.0002$, which is now seen to be an extremely accurate estimate.

Other biased methods tried include forming Padé approximants to the thrice differentiated series then raised to the power $(\frac{2}{3})$. Many approximants were defective, but those that were not gave steadily increasing estimates of μ^{-2}, so yielding $\mu^2 < 6.9607$.

Table 3. Estimates of μ^2 for the square lattice polygon generating function from the Levin transformed sequence of ratios. Biased estimates assume a critical exponent of $-\frac{3}{2}$.

	Unbiased				Biased			
n	$a_n^{(0)}$	$a_n^{(1)}$	$a_n^{(2)}$	$a_n^{(3)}$	$a_n^{(0)}$	$a_n^{(1)}$	$a_n^{(2)}$	$a_n^{(3)}$
5	4.741 935	9.213 790	7.239 213	6.520 660	9.483 871	7.004 695	6.909 411	6.948 877
6	4.996 599	8.827 329	7.123 822	7.005 481	8.565 598	6.967 804	6.927 282	6.969 233
7	5.196 732	8.352 835	7.083 897	6.723 806	8.083 806	6.954 013	6.944 636	6.945 490
8	5.359 314	8.045 016	6.957 396	7.031 530	7.795 365	6.951 138	6.944 949	6.963 566
9	5.494 244	7.797 591	6.976 776	6.935 756	7.607 415	6.949 431	6.951 332	6.958 400
10	5.608 156	7.634 006	6.966 314	6.972 115	7.477 542	6.949 912	6.953 547	6.960 986
11	5.705 607	7.513 284	6.967 643	6.947 781	7.383 727	6.950 760	6.955 722	6.959 212
12	5.789 937	7.423 546	6.963 337	6.958 678	7.313 605	6.951 835	6.956 669	6.959 913
13	5.863 634	7.353 761	6.962 404	6.957 365	7.259 737	6.952 812	6.957 495	6.959 778
14	5.928 592	7.298 798	6.961 456	6.958 725	7.217 416	6.953 698	6.958 042	6.959 861
15	5.986 281	7.254 649	6.960 973	6.958 706	7.183 538	6.954 472	6.958 453	6.959 841
16	6.036 858	7.218 662	6.960 595	6.959 131	7.155 979	6.955 141	6.958 750	6.959 864
17	6.084 244	7.188 924	6.960 363	6.959 269	7.133 252	6.955 715	6.958 976	6.959 866
18	6.126 187	7.164 065	6.960 199	6.959 391	7.114 282	6.956 208	6.959 148	6.959 869
19	6.164 296	7.143 067	6.960 084		7.098 280	6.956 631	6.959 281	
20	6.199 073	7.125 167			7.084 655	6.956 995		
21	6.230 938				7.072 956			
22	6.260 241				7.062 836			

Biased Neville table extrapolation for μ^2 (which implicitly assumes the absence of any confluent term) gave $\mu^2 = 6.9597 \pm 0.0009$, or $\mu = 2.638\,12 \pm 0.000\,17$.

In order to estimate the value of any possible confluent exponent Δ, we tried all the standard biased methods, with $\mu^2 = 6.959\,86$. The recurrence relation method gave no useful results, due to overflows in solving associated polynomial equations. The Baker-Hunter (1973) transformation was a little more successful, and gave weak evidence of a second exponent $\Delta = 0.75 \pm 0.16$. The Adler-Moshe-Privman method (1982) was instructive, and with $\Delta = 0.84$, estimates of α were very stable at $\alpha = 0.4997 \pm 0.0011$, where the quoted error is ± 1 standard deviation, based on a sample of the last nine table entries, and taking $\mu = 2.638\,155$. With $\mu = 2.638\,14$, and $\Delta = 0.835$, estimates of α were even better, at $\alpha = 0.5000 \pm 0.0008$. Repeating this analysis for the SAW series, for which we expect $\gamma = 43/32 = 1.343\,75$, we found $\gamma = 1.3434 \pm 0.0003$ with $\mu = 2.638\,155$ and $\Delta = 0.84$, and $\gamma = 1.3441 \pm 0.0003$ with $\mu = 2.638\,14$ and $\Delta = 0.835$.

This accords well with an earlier analysis of the triangular lattice SAW series (Guttmann 1984), in which we pointed out there was some evidence of a confluent exponent with a value of $\Delta = 0.84$.

Thus we conclude that the Adler-Moshe-Privman method *does* consistently indicate a correction to scaling exponent of $\Delta \approx 0.84$. It is unfortunate that no other method enables a similar conclusion to be drawn. Indeed, both Privman (1984) and Majid *et al* (1983) find evidence for $\Delta \approx 0.65$.

Turning now to the PGF for the Manhattan and L lattices, we observe that the small number of coefficients makes conventional methods of analysis difficult, in the sense that convergence is rather slow. We have therefore adopted an alternative technique, which relies on the assumption that the asymptotic form of the PGFs for the Manhattan and L lattices is the same as that for the square lattice. Thus if p_n^S, p_n^M and p_n^L denotes

the coefficients of x^n in the PGF for the square, Manhattan and L lattice respectively, we have

$$p^{S}_{4n} \sim K_S \mu_S^{4n}, \qquad p^{M}_{4n} \sim K_M \mu_M^{4n}, \qquad p^{L}_{4n} \sim K_L \mu_L^{4n}, \qquad n = 1, 2, 3, 4, \ldots, \tag{3.3}$$

and hence

$$\begin{aligned} r^{M}_{n} &= (p^{S}_{4n}/p^{M}_{4n})^{1/4n} \sim (\mu_S/\mu_M)(K_S/K_M)^{1/4n}, \qquad n = 1, 2, 3, 4, \ldots, \\ r^{L}_{n} &= (p^{S}_{4n}/p^{L}_{4n})^{1/4n} \sim (\mu_S/\mu_L)(K_S/K_L)^{1/4n}. \end{aligned} \tag{3.4}$$

To estimate μ_S/μ_M and μ_S/μ_L we form the sequences

$$\begin{aligned} s^{M}_{n} &= \exp\{4[n \log(r^{M}_{n}) - (n-1)\log(r^{M}_{n-1})]\}, \\ s^{L}_{n} &= \exp\{4[n \log(r^{L}_{n}) - (n-1)\log(r^{L}_{n-1})]\}. \end{aligned} \tag{3.5}$$

The sequences $\{s^{M}_{n}\}$, $\{s^{L}_{n}\}$ so obtained can be expected to converge in a manner determined by the confluent and analytic correction terms. We have found that the Levin u-transform gives a reasonably well converged sequence of estimates, and these results are shown in table 4, where we have taken $\{s^{M}_{n}\}$ and $\{s^{L}_{n}\}$ as $\{a^{(0)}_{n}\}$ in (3.2). From them we estimate

$$\mu_S/\mu_M = 1.5225 \pm 0.0004, \qquad \mu_S/\mu_L = 1.685 \pm 0.002. \tag{3.6}$$

Table 4. Estimates of μ for the Manhattan and L lattice polygon generating function as a multiple of μ for the square lattice.

	Manhattan				L lattice			
n	$a^{(0)}_{n}$	$a^{(1)}_{n}$	$a^{(2)}_{n}$	$a^{(3)}_{n}$	$a^{(0)}_{n}$	$a^{(1)}_{n}$	$a^{(2)}_{n}$	$a^{(3)}_{n}$
3	1.499 906	1.471 304	1.517 692	1.513 331	2.051 544	2.272 609	1.546 171	1.668 264
4	1.508 845	1.521 806	1.357 874	1.521 201	1.855 240	1.656 530	1.667 387	1.666 664
5	1.517 645	1.524 996	1.521 926	1.527 200	1.577 273	1.667 241	1.666 441	1.666 885
6	1.519 905	1.522 692	1.522 983	1.522 266	1.665 969	1.665 770	1.667 450	1.664 857
7	1.521 209	1.522 810	1.522 396	1.522 478	1.667 014	1.701 048	1.679 386	1.686 238
8	1.521 804	1.522 549	1.522 448	1.522 409	1.672 614	1.685 005	1.682 384	1.676 820
9	1.522 131	1.522 491	1.522 416		1.676 708	1.683 671	1.697 737	
10	1.522 294	1.522 438			1.679 144	1.685 058		
11	1.522 374				1.680 570			
12	1.522 407				1.681 553			

Similar though less well converged estimates are obtained from Neville table extrapolation (not shown). Using our estimate of $\mu_S = 2.638\,155$ we find

$$\mu_M = 1.7328 \pm 0.0005, \qquad \mu_L = 1.5657 \pm 0.0019. \tag{3.7}$$

These are in excellent agreement with the estimates (Guttmann 1983) based on the chain generating functions of the two series of $\mu_M = 1.7340 \pm 0.0015$ and $\mu_L = 1.5658 \pm 0.0010$.

In the preceding analysis for μ_M and μ_L, the discerning reader will have noticed that we require the coefficient p^{S}_{48}, and we have only obtained the square lattice PGF up to the coefficient in p^{S}_{46}. To estimate p^{S}_{48} we have used the recurrence relation method, discussed above, to predict the next coefficient from the available coefficients. In this way we estimate $p^{S}_{48} \approx 5\,814\,401\,613\,000\,000$ which we expect to be accurate to

the 10 quoted significant digits. This estimate of accuracy which follows from the prediction of known coefficients is rather conservative and is more than sufficient for the analysis of μ_M and μ_L.

4. Conclusion

We have substantially extended the PGF series for the square, L and Manhattan lattices, by using the finite lattice method and its extensions that allow directed lattices to be treated.

Analysis of the extended series allows accurate estimates of the connective constant to be made, particularly for the square lattice. For the three lattices we find

$$\mu_S = 2.638\,155 \pm 0.000\,025, \qquad \mu_L = 1.5657 \pm 0.0019, \qquad \mu_M = 1.7328 \pm 0.0005. \tag{4.1}$$

Our analysis also provides strong support for Nienhuis's (1982, 1984) results, which, coupled with scaling, imply that $\alpha = \frac{1}{2}$ exactly, and gives some evidence, unfortunately based on only one method, for a correction to scaling exponent of $\Delta \approx 0.84$. It would be very worthwhile if the square lattice SAW series could be extended by three or four terms in order to check this result.

In our earlier analysis (Guttmann 1984), a number of possible exact forms for μ (SQ) were given. The only one to survive our refined estimate is $\mu = \sqrt{11+\sqrt{5}-1} = 2.638\,140\ldots$, which we believe has the status of an interesting possibility, and useful mnemonic.

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